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Biofluiddynamics of balistiform and gymnotiform locomotion. Part 4. Short-wavelength limitations on momentum enhancement

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Elongated-body theory, used by Lighthill & Blake (1990) to investigate fish locomotion by undulatory movements of median fins, and to demonstrate momentum enhancement in the case when motile fins are attached to a rigid fish body of far greater depth, approximates local fluid motions by solutions of the two-dimensional Laplace equation. Here, a better local approximation (equation (2) below) to the three-dimensional Laplace equation for fluid motions of undulatory type is used to investigate the possibility of short-wavelength limitations on momentum enhancement. In an extreme case (fish bodies of very small width and very large depth) when on elongated-body theory the momentum enhancement factor β is predicted to become very large, short-wavelength considerations are shown to place a specific upper limit on β (see figure 2). In more general cases, this upper limit should perhaps be regarded as coexisting with other upper limits associated with *either* non-zero width *or* finite depth of fish body.

Short-wavelength limitations on momentum enhancement are of some biological interest as implying the existence not only of advantages (including a reduction in body drag) but also of some competing disadvantages (limitations in propulsive force) arising from progressive reductions in the wavelength of fin undulations.

1. Introduction

For the biological background to this analysis, see Lighthill & Blake (1990, hereafter referred to as Part 1). The present paper pursues a suggestion in §5 of Part 1 about the possibility of estimating to what extent reductions in the wavelength λ of undulatory motions of median fins attached to a rigid fish body may limit momentum enhancement.

Quite simply, this suggestion is that local enhancement of momentum, calculated in Part 1 for potential flows satisfying the two-dimensional Laplace equation

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0 \tag{1}$$

(in accordance with the principles of elongated-body theory), should be recalculated for solutions of the modified equation

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 - k^2\phi = 0 \quad (\text{where } k = 2\pi/\lambda) \tag{2}$$

which follows from the three-dimensional Laplace equation for a potential ϕ varying sinusoidally† in the z-direction with wavelength λ. Some reduction in the calculated momentum can be expected because the fundamental solution of (2) is exponentially attenuated with distance r like $\exp(-kr)$.

The present paper confirms such expectations through a calculation of the momentum enhancement factor β along these lines in a simple limiting case. This is the case when, whereas the *fin* depth

$$l-s=b \tag{3}$$

may have an arbitrary magnitude in comparison with

$$k^{-1}=\lambda/2\pi, \tag{4}$$

nevertheless, the overall depth of fish, $2l$, is large compared with k^{-1}. Our analysis takes the fish body width small (as in §4 of Part 1) but is equally applicable to balistiform and to gymnotiform locomotion.

The result (see §2) of this fairly simple calculation for $2lk$ large is shown in figure 2, where we see also that when at the same time kb is relatively small the momentum enhancement factor is closely approximated as

$$\beta = 1.53\ (kb)^{-\frac{1}{2}}. \tag{5}$$

Such results can be viewed as indicating yet one more restriction on that broad tendency for β to increase without limit as $s/l \rightarrow 1$ which is illustrated for small fish body width in Part 1, figure 6 (and for gymnotiform locomotion in Part 3 (Lighthill 1990), figure 4); here the limit $s/l \rightarrow 1$ is the same as the limit $b/l \rightarrow 0$. Admittedly the assumption of either a non-zero axis ratio t/s (width to depth) for the fish body cross-section or a non-zero value of b/l was found in Parts 3 and 1 to restrict β to finite values, shown to be around 3 for typical fish geometries; against this background, we have to ask whether the exponential attenuation of a fundamental solution of equation (2) may act so as to restrict the momentum enhancement factor β to upper limits still lower than the geometry of the fish cross-section would itself impose.

Equation (5) gives us a positive answer to this question unless the product kb is less than or equal to about $\frac{1}{4}$. Specifically, for thin fish bodies, the momentum generated by given movements of a median fin may be limited in value either by the finite depth of the fish body or (in cases of very large fish body depth) by the exponential attenuation, like $\exp(-kr)$, of the influence of the fin's movements. When kb is less than or equal to about $\frac{1}{4}$ the former limitation predominates, but for larger values of kb the exponential attenuation may more stringently limit β, to an upper bound given at most by equation (5); an equation of which this interpretation is more fully explained in §2.

From the biological point of view, the above conclusion raises certain rather interesting questions, related to the distinction between two main types – undulatory and oscillatory – of balistiform locomotion found (see Part 1, §2) in nature. The coexistence of two types of balistiform locomotion might be reckoned to imply competing benefits; yet elongated-body theory (Part 1) suggests that both share to a comparable extent the advantage of enhanced thrust that comes from momentum enhancement, while only the undulating type enjoys the benefit of reduction in body drag which results from the virtual elimination of lateral oscillations in sideslip and

† Admittedly, any periodic variation of ϕ in the z-direction is unlikely to be very closely sinusoidal; nevertheless, its leading Fourier component (which by definition is sinusoidal) can be expected to make the main contribution to the momentum in those 'configurations (a) and (c)' that are defined in figure 4 of Part 1.

yaw. Now we see that any evolutionary trends towards undulatory motion at relatively shorter wavelength, such as may have tended to win this drag-reduction benefit, will have carried with them a partly countervailing disadvantage: a gradual reduction of thrust enhancement. This must have acted, as suggested in Part 1, §5, to limit the advantage of further reduction of wavelength beyond a certain point.

2. Recalculation of momentum enhancement in the limiting case $2lk$ large

The case where the wavelength λ of fin undulation is very short compared with 2π times the overall depth $2l$ of the fish cross-section may be treated as the limiting case

$$2lk \to \infty. \tag{6}$$

Figure 1 shows the geometry of the fluiddynamic problem which needs to be solved in this limiting case: effectively, each fin (that is, each of the two in the case of balistiform locomotion, or the single median fin in the case of gymnotiform locomotion) moves as if attached to a thin rigid fish body of unlimited depth. The resulting fluid momentum M per unit length of fish must be calculated in order to determine the momentum enhancement factor

$$\beta = M/M_1. \tag{7}$$

Here, the denominator M_1 represents the momentum

$$M_1 = \tfrac{1}{8}\pi\rho\omega b^3 \tag{8}$$

associated in elongated-body theory (see equation (32) of Part 1) with the same motion of the fin in question 'on its own' in unbounded fluid.

If in the problem of figure 1 the velocity potential ϕ satisfied the two-dimensional Laplace equation (1), then the associated momentum M would be infinite (and indeed figure 6 of Part 1 shows how $\beta \to \infty$ as $s/l \to 1$; that is, as $b/l \to 0$). However, the use of the modified equation (2) for ϕ produces an exponential attenuation in the effect of the fin motions with distance from the fin, and this limits the momentum to a finite value.

We calculate M by the same method that was used in equations (4) to (10) of Part 1. Once again, we write

$$(\partial\phi/\partial y)_{y=0} = f(x); \tag{9}$$

a function which, in the problem of figure 1, takes the form

$$f(x) = \omega(b-x) \quad (0 < x < b), \quad 0 \quad (x > b), \tag{10}$$

The required momentum M per unit length can be expressed in the same form

$$M = \rho\oint\phi\,\mathrm{d}x \tag{11}$$

as in Part 1, equation (6), with the integral taken around the surface of the semi-infinite plate in the positive (anticlockwise) sense.

Again as before, we define ϕ_1 as the solution of (2) satisfying the boundary condition (9) with $f(x) = 1$; thus,

$$(\partial\phi_1/\partial y)_{y=0} = 1. \tag{12}$$

Here, we note that the famous diffraction study by Sommerfeld (1895) solved an analogous problem for the wave equation (that is, for (2) with $+k^2$ replacing $-k^2$)

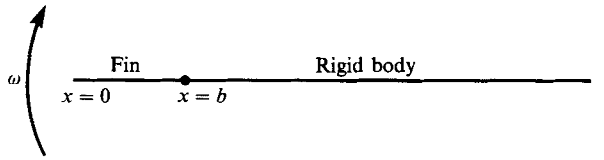

FIGURE 1. The fluiddynamical problem in the limiting case $2lk \to \infty$.

in terms of error functions of imaginary argument. A similar study in §3 below derives ϕ_1 in terms of the error function of real argument,

$$\mathrm{erf}(z) = 2\pi^{-\frac{1}{2}} \int_0^z \exp(-\zeta^2)\,\mathrm{d}\zeta. \tag{13}$$

In addition, the value of ϕ_1 on the plate is derived in a particularly simple form as

$$\phi_1 = \mp k^{-1}\,\mathrm{erf}\,[(kx)^{\frac{1}{2}}], \tag{14}$$

with the $\mp$ sign standing for $-$ on the upper surface and $+$ on the lower surface of the plate.

As in equation (8) of Part 1 we can use the property (12) of ϕ_1 to rewrite the momentum (11) as

$$M = \rho \oint \phi(\partial\phi_1/\partial y)\,\mathrm{d}x \tag{15}$$

and then use Green's theorem to re-express it as

$$M = \rho \oint \phi_1(\partial\phi/\partial y)\,\mathrm{d}x. \tag{16}$$

Note that Green's theorem continues to give this result although both ϕ and ϕ_1 satisfy (2) rather than (1); thus, the difference between the integral (15) and (16) can be written as ρ times the integral over the entire fluid region of

$$\phi(\partial^2\phi_1/\partial x^2 + \partial^2\phi_1/\partial y^2) - \phi_1(\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2) = k^2\phi\phi_1 - k^2\phi_1\phi = 0. \tag{17}$$

Finally, substituting from (9) and (14) in the integral (16) for the momentum, where the change in x is negative on the upper and positive on the lower surface, we obtain

$$M = 2\rho k^{-1} \int_0^\infty f(x)\,\mathrm{erf}\,[(kx)^{\frac{1}{2}}]\,\mathrm{d}x. \tag{18}$$

It is now straightforward to calculate M for the particular form of $f(x)$ shown in (10) as

$$M = \rho\omega k^{-3}\{(k^2b^2 - kb + \tfrac{3}{4})\,\mathrm{erf}\,[(kb)^{\frac{1}{2}}] + (\tfrac{1}{2}kb - \tfrac{3}{4})(kb)^{\frac{1}{2}}\,\mathrm{erf}'\,[(kb)^{\frac{1}{2}}]\} \tag{19}$$

in terms of the error function (13) and its first derivative. Equations (7), (8) and (19) allow us to plot the momentum enhancement factor β as a function of kb in figure 2, whose general significance has already been discussed.

The limiting behaviours of β as $kb \to 0$ and as $kb \to \infty$ deserve detailed explanation. Since $f(x) = 0$ for $x > b$, equation (18) can be well approximated for small kb if we

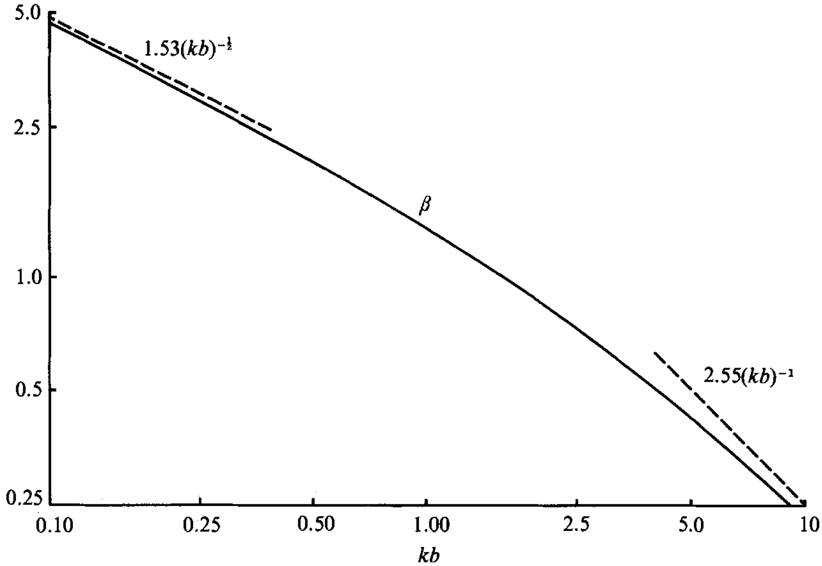

FIGURE 2. The momentum enhancement factor β plotted as a function of kb in the limiting case $2lk \to \infty$.

replace $\operatorname{erf} z$ in the definition (13) by its form $2\pi^{-\frac{1}{2}}z$ for small z. Then (18), with $f(x)$ as in (10), becomes

$$M = 4\rho(\pi k)^{-\frac{1}{2}}G, \tag{20}$$

with

$$G = \int_0^\infty x^{\frac{1}{2}}f(x)\,\mathrm{d}x \tag{21}$$

$$= \tfrac{4}{15}\omega b^{\frac{5}{2}}, \tag{22}$$

and this gives

$$\beta = (128/15\pi^{\frac{3}{2}})\,(kb)^{-\frac{1}{2}} = 1.53(kb)^{-\frac{1}{2}}. \tag{23}$$

An independent way of arriving at this result (20) for the momentum is to note that for kb small the motion in the vicinity of the fin itself will closely satisfy the two-dimensional Laplace equation (1). Accordingly, a simple conformal mapping can be used to determine the form of that motion and to show that its far-field representation is as

$$\phi \sim -2\pi^{-1}\,Gr^{-\frac{1}{2}}\cos\tfrac{1}{2}\theta \tag{24}$$

(in polar coordinates with $x = r\cos\theta$, $y = r\sin\theta$). This far-field representation – which, in the language of matched asymptotic expansions, is the outer limit of an inner solution – must be matched to an outer solution of (2) which has the property (24) in the inner limit as $kr \to 0$. This solution is

$$\phi = -2\pi^{-1}G\,\mathrm{e}^{-kr}\,r^{-\frac{1}{2}}\cos\tfrac{1}{2}\theta \tag{25}$$

as will be evident to anyone who remembers that the modified Bessel function of the second kind takes for order $\frac{1}{2}$ the value

$$K_{\frac{1}{2}}(z) = \mathrm{e}^{-z}(\pi/2z)^{\frac{1}{2}} \tag{26}$$

and that (2) is satisfied by

$$K_{\frac{1}{2}}(kr)\cos\tfrac{1}{2}\theta. \tag{27}$$

Finally, the integral (11) for the momentum M is calculated from the expression (25) for ϕ as

$$M = 4\rho\pi^{-1} G \int_0^\infty e^{-kx} x^{-\frac{1}{2}} \, dx = 4\rho G(\pi k)^{-\frac{1}{2}}. \tag{28}$$

This alternative way of deriving (20) has the merit of relating it directly to the tendency of a typical solution (25) to (2) to fall off exponentially like $\exp(-kr)$ as r increases.

In the other limit as kb becomes large the error function in (20) tends rapidly to 1, and its derivative to zero, so that

$$M \sim \rho\omega b^2 k^{-1} - \rho\omega b k^{-2} + \dots . \tag{29}$$

The leading term in (29) gives the result

$$\beta \sim 8\pi^{-1}(kb)^{-1} = 2.55\ (kb)^{-1} \tag{30}$$

displayed in figure 2. Such a result in the limit of extremely short wavelength can be interpreted in terms of that 'effective penetration depth' of k^{-1} on both sides of the plate which characterizes solutions of (2) such as

$$e^{-ky} \quad \text{for} \quad y > 0, \quad \text{or} \quad e^{+ky} \quad \text{for} \quad y < 0. \tag{31}$$

This gives an effective added mass

$$2\rho k^{-1} \, dx \tag{32}$$

for each element dx of fin subject to the motion (10); and, after integration, this makes the momentum

$$2\rho k^{-1} \int_0^b \omega(b-x) \, dx = \rho\omega b^2 k^{-1} \tag{33}$$

as in the leading term of (29).

On the other hand, figure 2 shows that β is not so closely represented by its leading term for large kb as is the case for small kb. This situation is associated with the negative second term in (29), which evidently reflects the fact that momentum in the positive y-direction above and below the fin in figure 2 is accompanied by momentum in the negative y-direction in an area of order k^{-2} (as $kb \to \infty$) just to the left of the fin tip.

This section's use of (2) to calculate β in a particular limiting case has valuably emphasized important restrictions on the accuracy of calculations based on the two-dimensional Laplace equation (1), as in elongated-body theory. Needless to say, however, these latter calculations remain significant for many purposes of comparisons between properties of different cross-sectional geometries.

3. Derivation of the ϕ_1 solution

In order to find the solution ϕ_1 of (2) subject to the boundary condition (12), we bear in mind the close analogy to Sommerfeld's diffraction problem, where

(i) the geometry is the same;
(ii) the partial differential equation is the same with the sign of k^2 changed;
(iii) the boundary condition for the scattered wave is the same.

The original treatment by Sommerfeld (1895) was simplified greatly by Lamb (1906) and a little more in Lamb (1932, article 308); and Lamb's method is here adapted to the present problem.

Bearing in mind the interest in solutions which are expected (especially for kx large and positive) to resemble (31), we consider alternative solutions of (2) in the forms

$$\phi = \mathrm{e}^{+ky}\phi_+ \quad \text{and} \quad \phi = \mathrm{e}^{-ky}\phi_-. \tag{34}$$

Then the equations satisfied by ϕ_+ and ϕ_- are

$$\partial^2\phi_\pm/\partial x^2 + \partial^2\phi_\pm/\partial y^2 \pm 2k\,\partial\phi_\pm/\partial y = 0, \tag{35}$$

where either the upper or the lower choice of sign is to be made throughout.

We now define parabolic coordinates by the equations

$$kx = \xi^2 - \eta^2, \quad ky = 2\xi\eta; \tag{36}$$

with $\eta > 0$ in the fluid and with the limits

$$\eta \to 0 \quad \text{for} \quad \xi > 0 \quad \text{and} \quad \eta \to 0 \quad \text{for} \quad \xi < 0 \tag{37}$$

corresponding to the upper and lower surfaces of the plate. In terms of ξ and η the equations (35) become

$$\partial^2\phi_\pm\partial\xi^2 + \partial^2\phi_\pm/\partial\eta^2 \pm 4(\eta\,\partial\phi_\pm/\partial\xi + \xi\,\partial\phi_\pm/\partial\eta) = 0; \tag{38}$$

which (as Lamb's treatment of the analogous problem suggests) have solutions

$$\phi_\pm = f_\pm(\xi+\eta) + g_\pm(\xi-\eta) \tag{39}$$

provided the functions $f_\pm(\zeta)$ and $g_\pm(\zeta)$ satisfy

$$f''_\pm(\zeta) \pm 2\zeta f'_\pm(\zeta) = 0, \quad g''_\pm(\zeta) \mp 2\zeta g'_\pm(\zeta) = 0. \tag{40}$$

Only the solutions f_+ and g_- of these equations show an acceptable avoidance of exponential growth as $|\zeta| \to \infty$. We take

$$\phi_+ = A\int_{\xi+\eta}^{\infty} \exp(-\zeta^2)\,\mathrm{d}\zeta, \quad \phi_- = B\int_{-\infty}^{\xi-\eta} \exp(-\zeta^2)\,\mathrm{d}\zeta, \tag{41}$$

involving solutions f_+ and g_- of (4) such that f_+ vanishes exponentially as $\zeta \to +\infty$ (and remains finite as $\zeta \to -\infty$) while g_- vanishes exponentially as $\zeta \to -\infty$ (and remains finite as $\zeta \to +\infty$). Since (36) for ξ and η imply that

$$(\xi+\eta)^2 = k(r+y), \quad (\xi-\eta)^2 = k(r-y) \quad \text{with} \quad r^2 = x^2+y^2 \tag{42}$$

the corresponding solutions (34) for ϕ vanish exponentially both for large positive y (like e^{-kr} and e^{-ky} respectively) and for large negative y (like e^{+ky} and e^{-kr} respectively); as might be expected in terms of ideas of exponential attenuation proceeding either with distance r from the origin or with distance $|y|$ from the plate.

The boundary conditions satisfied on $\eta = 0$ (where $y = 0$) by these two solutions are

$$[\partial(\mathrm{e}^{+ky}\phi_+)/\partial y]_{y=0} = A\left[k\int_{\xi}^{\infty} \exp(-\zeta^2)\,\mathrm{d}\zeta - \tfrac{1}{2}k\xi^{-1}\exp(-\xi^2)\right] \tag{43}$$

and

$$[\partial(\mathrm{e}^{-ky}\phi_-)/\partial y]_{y=0} = B\left[-k\int_{-\infty}^{\xi} \exp(-\zeta^2)\,\mathrm{d}\zeta - \tfrac{1}{2}k\xi^{-1}\exp(-\xi^2)\right] \tag{44}$$

respectively; so that, if $A = B$, then the difference

$$\phi_1 = \mathrm{e}^{+ky}\phi_+ - \mathrm{e}^{-ky}\phi_- \tag{45}$$

has

$$(\partial\phi_1/\partial y)_{y=0} = Ak\int_{-\infty}^{\infty} \exp(-\zeta^2)\,\mathrm{d}\zeta = Ak\pi^{\frac{1}{2}}, \tag{46}$$

which is independent of ξ. Thus the choice

$$A = B = k^{-1}\pi^{-\frac{1}{2}} \tag{47}$$

makes ϕ_1 satisfy the correct boundary condition (12).

We deduce from (31), (45) and (47) that

$$\phi_1 = k^{-1}\pi^{-\frac{1}{2}}\left[\mathrm{e}^{ky}\int_{\xi+\eta}^{\infty}\exp\left(-\zeta^2\right)\mathrm{d}\zeta - \mathrm{e}^{-ky}\int_{-\infty}^{\xi-\eta}\exp\left(-\zeta^2\right)\mathrm{d}\zeta\right], \tag{48}$$

where $\xi+\eta$ and $\xi-\eta$ satisfy (42). In terms of the error function (13), we can write

$$\phi_1 = \tfrac{1}{2}k^{-1}\,\mathrm{e}^{ky}\left[1-\operatorname{erf}(\xi+\eta)\right] - \tfrac{1}{2}k^{-1}\,\mathrm{e}^{-ky}\left[1+\operatorname{erf}(\xi-\eta)\right]. \tag{49}$$

In particular, the value of ϕ_1 on $\eta = 0$ (where $y = 0$) can be written

$$\phi_1 = -k^{-1}\operatorname{erf}(\xi), \tag{50}$$

which agrees with the result in (14), on which §2 is based, since ξ is the positive or negative square root of kx on the upper or lower edge of the plate respectively.

It is a matter for satisfaction that this elegant error-function solution has made possible the modelling of balistiform and gymnotiform locomotion in yet another relevant limiting case.

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